

# Perturbed Markov Chains

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## Abstract

We study irreducible time-homogenous Markov chains with finite state space in discrete time. We obtain results on the sensitivity of the stationary distribution and other statistical quantities with respect to perturbations of the transition matrix. We define a new closeness relation between transition matrices, and use graph-theoretic techniques, in contrast with the matrix analysis techniques previously used.

**Keywords:** Markov chains, stationary distribution, exit distribution, conductance, sensitivity analysis, Perturbation theory, stability of a Markov chain.

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# 1 Introduction

The present paper concerns irreducible time-homogenous Markov chains with a finite state space in discrete time. We are interested in the effects of perturbations of the transition matrix on the stationary distribution and on other statistical quantities.

This question has a long history, starting with Schweizer [?]. Let  $S$  be a finite set, and  $q = (q(t | s))_{s,t \in S}$  an irreducible transition matrix. Let  $\hat{q} = ((\hat{q}(t | s))_{s,t \in S})$  be another transition matrix. Denote by  $\mu = (\mu_s)_{s \in S}$  and  $\hat{\mu} = (\hat{\mu}_s)_{s \in S}$  the stationary distributions that correspond to  $q$  and  $\hat{q}$  respectively. Schweizer [?] estimated  $\|\mu - \hat{\mu}\|_\infty$  as a function of  $\|q - \hat{q}\|_\infty$ , using the fundamental matrix of Kemeny and Snell [?]. A vast literature has followed up Schweizer, and provided various estimates to  $\|\mu - \hat{\mu}\|_\infty$  and  $\left| \frac{\mu_s - \hat{\mu}_s}{\mu_s} \right|$  ( $s \in S$ ) as a function of  $\|q - \hat{q}\|_\infty$ . See, e.g., [?], [?], [?], [?], [?], [?].

O’Cinneide [?] studied the effects of entry-wise relative perturbations on the stationary distribution; that is, he provided a bound on  $\left| \frac{\mu_s - \hat{\mu}_s}{\mu_s} \right|$  as a function of  $\max_{s,t \in S} \left\{ \frac{q(t|s)}{\hat{q}(t|s)}, \frac{\hat{q}(t|s)}{q(t|s)} \right\}$ .

Our paper differs from the existing literature in three respects. First, we introduce a new way to measure the difference between two transition matrices. In the spirit of the entry-wise relative perturbations of O’Cinneide [?], our measure is  $\max \left\{ \left| 1 - \frac{\hat{q}(t|s)}{q(t|s)} \right| \right\}$ , but the maximum is not taken over *all* pairs  $s, t \in S$ . Rather, the maximum is taken over all *frequent* transitions  $s \rightarrow t$ . Formally, the notion of closeness between  $q$  and  $\hat{q}$  is the following. For a transition matrix  $q$ , we define  $\zeta_q = \min_{\emptyset \subset C \subset S} \sum_{s \in C} \mu_s q(\overline{C} | s)$ , where  $\overline{C} = S \setminus C$  is the complement of  $C$  in  $S$ . This is a variant of the conductance (see, e.g., [?], [?], [?]). Given  $\varepsilon, \beta > 0$ , we say that  $\hat{q}$  is  $(\varepsilon, \beta)$ -close to  $q$  if for every two states  $s, t \in S$ ,  $\left| 1 - \frac{\hat{q}(t|s)}{q(t|s)} \right| \leq \beta$  whenever (a)  $\mu_s q(t|s) \geq \varepsilon \zeta_q$  or (b)  $\mu_s \hat{q}(t|s) \geq \varepsilon \zeta_q$ . Condition (a) holds whenever the transition from  $s$  to  $t$  occurs frequently. Condition (b) is not analogous to (a), since it involves the stationary distribution of  $q$ , and the transition matrix  $\hat{q}$ .

Provided  $\varepsilon$  and  $\beta$  are small enough, we show that  $\hat{q}$  is irreducible, and  $\frac{\mu_s}{\hat{\mu}_s}$  is close to 1 for each  $s \in S$ .

The logic behind this closeness relation, and its novelty, is that even large perturbations that occur in rarely visited states should not affect too much the stationary distribution. This point is illustrated in the example that is studied in the next section.

A motivation for using this closeness measure is the following statistical implication. Assume a statistician observes a realization  $s_1, \dots, s_N$  of (the first  $N$  components of) a Markov chain with unknown irreducible transition matrix  $q$ . Given these observations, he computes the empirical transition matrix  $\hat{q}$  and the invariant distribution  $\hat{\mu}$  of  $\hat{q}$  (if  $s_N = s_1$ ,  $\hat{\mu}$  coincides with the empirical occupancy measure). For fixed  $\varepsilon, \beta > 0$ , if  $N$  is large enough,  $\hat{q}$  is  $(\varepsilon, \beta)$ -close to  $q$  with probability close to one, hence  $\hat{\mu}$  is an accurate estimate of the invariant distribution of the underlying Markov chain.

Second, instead of using matrix analysis, we use the graph techniques developed by Freidlin and Wenzell [?] and extensively used in the analysis of Markov chains with rare transitions (see, e.g., Catoni [?]).

Third, our approach allows us to estimate the sensitivity of other statistical quantities, such as the exit distribution from a given set and the average length of visits to a given set.

The paper is organized as follows. Section 2 contains the statements of the main results. We also study there an example that illustrates the advantage of our closeness relation, and compares the bound we derive to existing bounds. Section 3 briefly recalls standard formulas, and states few elementary properties of graphs. Section 4 is devoted to the proof of the main result. The last section deals with a variation of the main result.

## 2 Notations and results

Let  $S$  be a finite set, fixed through the paper, with at least two elements. For every subset  $C \subseteq S$ ,  $\overline{C} = S \setminus C$  is the complement of  $C$  in  $S$ ,  $|C|$  is its cardinality, and  $\Delta(C)$  is the set of probability distributions over  $C$ . For  $s \in C \subseteq S$ , we write  $C \setminus s$  instead of the more cumbersome  $C \setminus \{s\}$ .

## 2.1 Main result

Let  $q$  be an irreducible transition matrix over  $S$ , with stationary distribution  $\mu = (\mu_s)_{s \in S}$ . For every  $C \subseteq S$  we denote  $\mu_C = \sum_{s \in C} \mu_s$ . Let  $\hat{q}$  be another transition matrix over  $S$ . Assuming  $\hat{q}$  is irreducible, we wish to bound the distance between  $\mu$  and  $\hat{\mu}$ .

Our notion of closeness of  $\hat{q}$  to  $q$  involves a measure of how mixing  $q$  is. Our measure involves the quantity

$$\zeta_q = \min_{\emptyset \subset C \subset S} \sum_{s \in C} \mu_s q(\overline{C} \mid s), \quad (1)$$

which is a variant of the *conductance*, see e.g. [?], [?], [?]. Given  $C \subset S$ , the quantity  $\sum_{s \in C} \mu_s q(\overline{C} \mid s)$  measures the average frequency of transitions out of  $C$ . Hence,  $\zeta_q$ , being the lowest such frequency, is a measure of how isolated a subset  $C$  may be. Formally,

**Definition 1** *Let  $\varepsilon, \beta > 0$ . We say that a transition matrix  $\hat{q}$  is  $(\varepsilon, \beta)$ -close to  $q$  if for every two states  $s, t \in S$ ,  $\left|1 - \frac{\hat{q}(t \mid s)}{q(t \mid s)}\right| \leq \beta$  whenever (a)  $\mu_s q(t \mid s) \geq \varepsilon \zeta_q$  or (b)  $\mu_s \hat{q}(t \mid s) \geq \varepsilon \zeta_q$ .*

Note that this closeness notion is not symmetric, since we use only the stationary distribution of  $q$ .

Denote  $L = \sum_{n=1}^{|S|-1} \binom{|S|}{n} n^{|S|}$ . We now state our main result.

**Theorem 2** *Let  $\beta \in (0, 1/2^{|S|})$  and let  $\varepsilon \in (0, \frac{\beta(1-\beta)}{L \times |S|^4})$ . For every irreducible transition matrix  $q$  on  $S$  and every transition matrix  $\hat{q}$  that is  $(\varepsilon, \beta)$ -close to  $q$ :*

1.  $\hat{q}$  is irreducible.
2. Its stationary distribution  $\hat{\mu}$  satisfies  $\left|1 - \frac{\hat{\mu}_s}{\mu_s}\right| \leq 18\beta L$  for each  $s \in S$ .

## 2.2 An example and comparison with existing bounds

As mentioned in the introduction, many authors provided bounds for the sensitivity of the stationary distribution.

We now study an example that first highlights the logic behind the closeness relation, and second shows that in some cases, the bound we give is better than existing bounds.

Fix  $\delta \in (0, 1/2)$ . Take a Markov chain with three states and transition matrix as follows.

$$q(2 | 1) = 1 - \delta, \quad q(3 | 1) = \delta, \quad q(1 | 2) = q(1 | 3) = 1.$$

Thus, in every other stage the process visits state 1. In particular, the stationary distribution  $\mu$  is given by

$$\mu_1 = \frac{1}{2}, \quad \mu_2 = \frac{1 - \delta}{2}, \quad \mu_3 = \frac{\delta}{2}.$$

The quantity  $\zeta_q$  is given by

$$\zeta_q = \min \left\{ \frac{1}{2}, \frac{1 - \delta}{2}, \frac{\delta}{2}, \frac{1}{2} \times \delta, \frac{1}{2} \times (1 - \delta), \frac{1 - \delta}{2} + \frac{\delta}{2} \right\} = \frac{\delta}{2}.$$

Fix  $0 < \beta < 1/8 = 1/2^3$ ,  $0 < \varepsilon < \beta(1 - \beta)/3^4 L$  and  $0 < \eta < \varepsilon$ . In particular, one can choose  $\eta = \beta/C$ , where  $C > 1$  is some fixed scalar, independent of  $\delta$ .

Define a transition matrix  $\hat{q}$  by

$$\hat{q}(2 | 1) = 1 - \delta, \quad \hat{q}(3 | 1) = \delta, \quad \hat{q}(1 | 2) = 1, \quad \hat{q}(1 | 3) = 1 - \eta, \quad \hat{q}(2 | 3) = \eta.$$

Thus, we only change the transitions out of state 3. Moreover, whereas the relative size of the transition  $3 \rightarrow 1$  changed moderately, the relative size of the transition  $3 \rightarrow 2$  changed dramatically.

We first argue that  $\hat{q}$  is  $(\varepsilon, \beta)$ -close to  $q$ . Indeed,  $\left| 1 - \frac{\hat{q}(t|s)}{q(t|s)} \right| = 0$  whenever  $s \neq 3$ , and  $\left| 1 - \frac{\hat{q}(1|3)}{q(1|3)} \right| = \eta < \beta$ . The claim now follows, since  $\mu_3 q(2 | 3) = 0$  and  $\mu_3 \hat{q}(2 | 3) = \frac{\delta}{2} \times \eta < \zeta_q \times \varepsilon$ .

Thus, Theorem 2 states that  $\left| \frac{\mu_3 - \hat{\mu}_3}{\mu_3} \right| \leq 18L\beta = 18LC\eta$ .

This example highlights the logic behind our closeness relation. Since state 3 is rarely visited, the stationary distribution is not too sensitive to changes in transitions out of this state, even though these changes are relatively large.

Cho and Meyer [?] bound the sensitivity of the stationary distribution by the mean first passage time:

$$\left| \frac{\mu_s - \hat{\mu}_s}{\mu_s} \right| \leq \frac{\|q - \hat{q}\|_\infty}{2} \times \max_{t \neq s} M_{t,s}, \quad (2)$$

where  $M_{t,s}$  is the mean first passage time from state  $t$  to state  $s$ .

Observe that in the example  $M_{2,3} = 2/\delta$  while  $M_{1,3} = 2/\delta - 1$ , hence the bound provided in [?] for  $\left| \frac{\mu_3 - \hat{\mu}_3}{\mu_3} \right|$  is  $\eta \times \frac{1}{\delta}$ , which is worse than our bound when  $\delta$  is small.

In the case of two-state chains, the bound (2) is very close to the bound that can be derived from Theorem 2, up to a universal constant. Indeed, (2) then reduces to

$$\left| \frac{\mu_s - \hat{\mu}_s}{\mu_s} \right| \leq \frac{1}{2} \max \{ |q(2|1) - \hat{q}(2|1)|, |q(1|2) - \hat{q}(1|2)| \} \times \max_{(3)} \left\{ \frac{1}{q(2|1)}, \frac{1}{q(1|2)} \right\}.$$

On the other hand,  $\zeta_q = \min \{ \mu_1 q(2|1), \mu_2 q(1|2) \}$ . Hence, Theorem 2 yields

$$\left| \frac{\mu_s - \hat{\mu}_s}{\mu_s} \right| \leq 36\beta, \text{ with } \beta = \max \left\{ \left| \frac{q(2|1) - \hat{q}(2|1)}{q(2|1)} \right|, \left| \frac{q(1|2) - \hat{q}(1|2)}{q(1|2)} \right| \right\},$$

provided  $\beta < 1/4$  - an inequality which is slightly more precise than (3), up to a constant 72.

Kirkland et al [?] bound the relative sensitivity by

$$\left| \frac{\mu_s - \hat{\mu}_s}{\mu_s} \right| \leq \frac{1}{2} \frac{\|q - \hat{q}\|_\infty}{2} \times \|A_s^{-1}\|_\infty,$$

where  $A$  is the fundamental matrix of [?], and  $A_s$  is the  $(n-1) \times (n-1)$  submatrix of  $A$  obtained by deleting the  $s$ 'th row and column.

One can verify that in our example  $A_3 = \begin{pmatrix} 1 & -1 + \delta \\ -1 & 1 \end{pmatrix}$ , so that  $A_3^{-1} = \begin{pmatrix} 1/\delta & -1 + 1/\delta \\ 1/\delta & 1/\delta \end{pmatrix}$ . Thus, the bound for  $\left| \frac{\mu_3 - \hat{\mu}_3}{\mu_3} \right|$  is  $\eta/2\delta$ , which is worse than our bound when  $\delta$  is small.

Note that the entry-wise ratio bound given by O'Cinneide [?] is not useful in this example, since  $q(2|3) = 0$  while  $\hat{q}(2|3) > 0$ .

### 2.3 The transition matrix changes in a subset of states

In some cases, the transition matrix is perturbed only in a subset  $S_1 \subset S$ , and restricted to  $S_1$  the transition matrix is sufficiently mixing, in the sense that the probability to reach any state in  $S_1$  before leaving  $S_1$  is bounded from below. In this case, instead of taking the conductance in the definition of the closeness relation, one can take another quantity, which is, in a sense, the conductance restricted to  $S_1$ .

Such a case occurs, for example, if the state space can be partitioned into some subsets, the transition matrix is mixing in each subset while the probability to move from one subset to another is small, and one perturbs the transition matrix only in one of the subsets.

Let  $S_1$  be a subset of  $S$ , with  $|S_1| > 1$ . Define

$$\zeta_q^1 = \min_{\emptyset \neq C \subset S_1} \sum_{s \in C} \mu_s q(\overline{C} \mid s).$$

Let  $(\mathbf{s}_n)$  be a Markov chain with transition matrix  $q$ . We denote by  $\mathbf{P}_{s,q}$  the law of  $(\mathbf{s}_n)$  when the initial state is  $s$ , and by  $\mathbf{E}_{s,q}$  the corresponding expectation.

For every proper subset  $C$  of  $S$  we let  $T_C = \min \{n \geq 0, \mathbf{s}_n \in C\}$  denote the first hitting time of  $C$  and  $T_C^+ = \min \{n \geq 1, \mathbf{s}_n \in C\}$  the first return to  $C$ . By convention, the minimum over an empty set is  $+\infty$ .

**Definition 3** Let  $\varepsilon, \beta > 0$ . We say that a transition matrix  $\hat{q}$  is  $(\varepsilon, \beta)$ -close to  $q$  on  $S_1$  if for every two states  $s, t \in S$ ,  $\left|1 - \frac{\hat{q}(t|s)}{q(t|s)}\right| \leq \beta$  whenever (a)  $\mu_s q(t|s) \geq \varepsilon \zeta_q^1$  or (b)  $\mu_s \hat{q}(t|s) \geq \varepsilon \zeta_q^1$ .

We now state the Theorem that corresponds to Theorem 2.

**Theorem 4** Let  $\beta \in (0, 1/2^{|S|})$ ,  $a > 0$  and  $\varepsilon \in (0, \frac{1}{2} \left(\frac{a}{L}\right)^{|S|} \times \frac{\beta(1-\beta)}{L \times |S|^4})$ . Let  $q$  be an irreducible transition matrix such that  $\mathbf{P}_{s,q}(T_{S_1 \cup \{t\}}^+ = T_{\{t\}}^+) \geq a$  for every  $s, t \in S_1$ . Then, for every transition matrix  $\hat{q}$  that is  $(\varepsilon, \beta)$ -close to  $q$  on  $S_1$  and that coincides with  $q$  on  $S \setminus S_1$ , we have

1. All states of  $S_1$  belong to the same recurrent set  $R$  for  $\hat{q}$ .

2. The stationary distribution  $\hat{\mu}$  of  $\hat{q}$  on  $R$  satisfies

$$\left| 1 - \frac{\hat{\mu}(s|S_1)}{\mu(s|S_1)} \right| \leq 18\beta L, \text{ for each } s \in S_1, \quad (4)$$

where  $\mu(s | S_1) = \mu_s / \mu_{S_1}$ .

Note that the claims in Theorem 4 differ from those in Theorem 2. It is no longer claimed that  $\hat{q}$  is irreducible, nor that the unconditional stationary distributions  $\mu$  and  $\hat{\mu}$  are close. The statements in Theorem 4 are optimal in this respect. This is due to the fact that the quantity  $\zeta_q^1$  contains no information on the frequency of transitions out of  $S_1$ . To emphasize this point, consider the following example.

Assume that  $S = \{a, b, c\}$  and  $S_1 = \{a, b\}$ . Let  $\varepsilon, \beta \in (0, 1/2)$  be given. Let two additional parameters  $\lambda$  and  $\eta$  be given in  $(0, 1)$ , and define  $q$  as follows. From state  $a$  (resp.  $b$ ) a chain with transition matrix  $q$  moves to  $c$  with probability  $\eta$ , and otherwise to  $b$  (resp. to  $a$ ). From state  $c$ , the chain remains in  $c$  with probability  $1 - \lambda$ , and otherwise moves to  $a$  or  $b$  with equal probability  $\frac{1}{2}\lambda$ .

Plainly,  $q$  is irreducible, and the value of  $\mu_a = \mu_b$  depends on the ratio  $\lambda/\eta$ : this common value may be arbitrary close to 0 (resp. to  $1/2$ ) provided  $\lambda/\eta$  is close enough to 0 (resp. to  $+\infty$ ). Note that  $\zeta_q^1 = \mu_a q(\{b, c\} | a) = \mu_a$ . Let now  $\hat{q}$  be defined exactly as  $q$ , except that the parameter  $\eta$  is replaced by another parameter  $\hat{\eta} \in [0, 1]$ . As soon as  $\eta, \hat{\eta} < \min\{\varepsilon, \beta\}$ ,  $\hat{q}$  is  $(\varepsilon, \beta)$ -close to  $q$ . This is in particular the case if  $\hat{\eta} = 0$ , in which case  $\hat{q}$  fails to be irreducible. On the other hand, even if  $\hat{\eta} > 0$ , the values of  $\eta, \hat{\eta}$  and  $\lambda$  can be chosen in such a way that the inequalities  $\eta, \hat{\eta} < \min\{\varepsilon, \beta\}$  are satisfied, and  $\eta \ll \lambda \ll \hat{\eta}$ . Hence, even if  $\hat{q}$  is irreducible, its unconditional stationary distribution  $\hat{\mu}$  may be arbitrarily far from  $\mu$ .

## 2.4 Sensitivity of other quantities

Our graph-theoretic approach allows us to obtain information on other quantities of interest. We here present the statements of the corresponding re-



sults.

We let  $\mathbf{Q}_{s,q}(\cdot|C)$  denote the law of the exit state from  $C$ :  $\mathbf{Q}_{s,q}(t|C) = \mathbf{P}_{s,q}(T_{\overline{C}} = T_t)$  for  $t \notin C$ . Next, we set

$$\begin{aligned} \nu_C(s) &:= \frac{\sum_{t \in \overline{C}} \mu_t q(s|t)}{\sum_{t \in \overline{C}} \mu_t q(C|t)} \text{ for } C \subset S \text{ and } s \in C, \text{ and} \\ K_C &:= \sum_{s \in C} \nu_C(s) \mathbf{E}_{s,q}[e_C] \text{ for } C \subset S. \end{aligned} \quad (5)$$

The numerator (resp. the denominator) in (5) is the long run frequency of transitions from  $\overline{C}$  to  $s$  (resp. from  $\overline{C}$  to  $C$ ). Thus,  $\nu_C(s)$  is the probability that the first stage in  $C$  the process visits is  $s$ , while  $K_C$  is the average length of a visit to  $C$ .

Assuming  $\hat{q}$  is irreducible, the corresponding quantities for  $\hat{q}$  will be denoted by  $\mathbf{Q}_{s,\hat{q}}$ ,  $\hat{\nu}_C(s)$  and  $\hat{K}_C$ . We now state the results on  $\mathbf{Q}_{s,\hat{q}}$  and  $\hat{K}_C$  that hold in the framework of Theorems 2 and 4 respectively.

**Theorem 5** *Set  $c = 2|S|^2$ . Under the assumptions of Theorem 2, the following holds: for each  $C \subset S$ ,*

1.  $\|\mathbf{Q}_{s,q}(\cdot|C) - \mathbf{Q}_{s,\hat{q}}(\cdot|C)\| < 12\beta L$  for every  $s \in C$
2.  $\frac{1}{c}K_C \leq \hat{K}_C \leq cK_C$ .

**Theorem 6** *Set  $c = 2|S|^2$ . Under the assumptions of Theorem 4, the following holds: for each  $C \subset S_1$ ,*

1.  $\|\mathbf{Q}_{s,q}(\cdot|C) - \mathbf{Q}_{s,\hat{q}}(\cdot|C)\| < 12\beta L$  for every  $s \in C$
2.  $\frac{1}{c}K_C \leq \hat{K}_C \leq cK_C$
3.  $\frac{1}{c}K_{S_1} \leq \hat{K}_{S_1} \leq cK_{S_1}$  or  $K_{S_1}, \hat{K}_{S_1} \geq \frac{1}{2\epsilon|S|} \times \frac{\mu_{S_1}}{\zeta_q^1}$ .

We let  $q$  be an irreducible transition matrix over  $S$ . It is fixed throughout the paper.

### 3 Preliminaries

Our computations are based on formulas due to Freidlin and Wenzell [?], that express stationary distribution, exit distributions and expected hitting times in graph-theoretic terms. For a discussion of some applications, we refer to Catoni [?]. These tools have also been used in the context of stochastic games in [?] and [?].

The weight of a graph is obtained from the transition probabilities corresponding to the different edges of the graph. We recall these formulas in section 3.1. Next, we compare the weights of a given graph under a transition matrix  $\hat{q}$  that is close to  $q$ .

#### 3.1 Reminder

Given  $C \subset S$ , a  $C$ -graph is a directed graph without cycle  $g$  over  $S$  such that:<sup>1</sup>

- For  $s \in C$ , there is exactly one edge starting at  $s$ , denoted by  $(s, g(s))$ .
- For  $s \in \overline{C}$ , there is no edge starting at  $s$ .

Thus, given  $s \in C$ , there is a unique path starting at  $s$  and ending at some  $t \in \overline{C}$ . We say that  $s$  leads to  $t$  along  $g$ . We denote by  $G(C)$  the set of  $C$ -graphs; for  $s \in C, t \in \overline{C}$ ,  $G_{s,t}(C)$  is the subset of graphs  $g \in G(C)$  such that  $s$  leads to  $t$  along  $g$ . Note that  $G(C)$  depends only on  $C$ , and not on the transition matrix. Note also that  $L$  bounds the number of graphs:  $L \geq \sum_{\emptyset \subset C \subset S} |G(C)|$ .

We identify each  $C$ -graph  $g$  with the collection of its edges:  $g = \cup_{s \in C} \{(s, g(s))\}$ .

Given  $D \subseteq C$ , and  $g \in G(C)$ , the *restriction of  $g$  to  $D$*  is defined to be the subgraph of  $g$  that contains exactly those edges of  $g$  that start in  $D$ . Thus, it is the  $D$ -graph  $g' = \cup_{s \in D} \{(s, g(s))\}$ .

For every  $g \in G(C)$ , we define the *weight* of  $g$  under  $q$  by

$$p(g) := \prod_{(s,t) \in g} q(t|s).$$

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<sup>1</sup>Our  $C$ -graphs correspond to  $\overline{C}$ -graphs in [?], [?].

The following formulas were derived by Freidlin and Wentzell [?], Lemmas 6.3.1, 6.3.4 and 6.3.3. For more direct statements and alternative proofs see Catoni [?].

**Proposition 7 (Freidlin-Wenzell, 1984)** *Let  $(S, q)$  be a Markov chain.*

- *If  $q$  is irreducible then for every  $s \in S$*

$$\mu_s = \frac{\sum_{G(S \setminus \{s\})} p(g)}{\sum_{y \in S} \sum_{G(S \setminus \{y\})} p(g)}. \quad (6)$$

- *For every proper subset  $C$  of  $S$  and every  $s \in C$ ,*

$$\mathbf{E}_{s,q} [T_{\overline{C}}] = \frac{\sum_{G(C \setminus \{s\})} p(g) + \sum_{t \in C, t \neq s} \sum_{G_{s,t}(C \setminus \{t\})} p(g)}{\sum_{G(C)} p(g)}, \quad (7)$$

*and*

$$\mathbf{Q}_{s,q}(t|C) = \frac{\sum_{G(C)} p(g)}{\sum_{G_{s,t}(C)} p(g)} \text{ for each } t \notin C. \quad (8)$$

### 3.2 Basic properties

In this section we provide basic properties of weights of graphs. The transition matrix  $q$  is here arbitrary.

**Definition 8** *Let  $C$  be a proper subset of  $S$ , and let  $\eta > 0$ . A graph  $g \in G(C)$  is  $\eta$ -maximal if*

$$p(g) \geq \eta \max_{g' \in G(C)} p(g').$$

We denote by  $G^\eta(C)$  the set of  $\eta$ -maximal  $C$ -graphs. For simplicity of notations, we do not emphasize the dependency of  $G^\eta(C)$  on the transition matrix. Clearly,  $G^\eta(C)$  is non-empty, for every  $\eta \leq 1$  and  $C \subset S$ . It is worth listing a few basic properties of graphs that we use repeatedly.

**Proposition 9** ***P0** Let  $C_1 \cap C_2 = \emptyset$ , and  $g_i \in G(C_i)$ , for  $i = 1, 2$ . If all paths of  $g_1$  lead to  $\overline{C_1 \cup C_2}$ , then  $g_1 \cup g_2$  is a  $C_1 \cup C_2$ -graph.*

***P1** Let  $C_1 \cap C_2 = \emptyset$ ,  $g \in G^\eta(C_1 \cup C_2)$ , and  $g_i$  the restriction of  $g$  to  $C_i$ . If all paths of  $g_2$  lead to  $\overline{C_1 \cup C_2}$ , then  $g_1 \in G^\eta(C_1)$ .*

**P2** Let  $C_1 \cap C_2 = \emptyset$ , and  $g_i \in G^{\eta_i}(C_i)$  for  $i = 1, 2$ . If  $g_1 \cup g_2$  is a  $C_1 \cup C_2$ -graph, then it is  $\eta_1 \eta_2$ -maximal.

**Proof.** **P0** and **P2** follow from the definitions. We now show that **P1** holds. Otherwise, there is  $g'_1 \in G(C_1)$  such that  $p(g_1) < \eta p(g'_1)$ . By **P0**,  $g' = g'_1 \cup g_2$  is in  $G(C_1 \cup C_2)$ , but  $p(g) < \eta p(g')$ , a contradiction. ■

Note that **P1** needs not hold without the condition that all paths of  $g_2$  lead to  $\overline{C_1 \cup C_2}$ . Indeed, take  $S = \{1, 2, 3, 4\}$ ,  $C_1 = \{1\}$ ,  $C_2 = \{2\}$ , and  $q(2 \mid 1) = q(1 \mid 2) = 1 - q(3 \mid 1) = 1 - q(4 \mid 2) = 2/3$ . The  $C_2$ -graph  $g_1 = (2 \rightarrow 4)$  is  $1/2$ -maximal, and the  $C_1 \cup C_2$ -graph  $(1 \rightarrow 2, 2 \rightarrow 4)$  is 1-maximal.

**Lemma 10** Let  $C$  be a proper subset of  $S$ , let  $\eta > 0$ , and let  $H$  be a set of graphs such that  $G^\eta(C) \subseteq H \subseteq G(C)$ . Then

$$0 \leq \frac{\sum_{g \in G(C)} p(g)}{\sum_{g \in H} p(g)} - 1 < \eta L.$$

In particular,

$$0 \leq 1 - \frac{\sum_{g \in H} p(g)}{\sum_{g \in G(C)} p(g)} < \eta L.$$

**Proof.** Since  $H \subseteq G(C)$ , and by the definition of  $G^\eta(C)$ ,

$$0 \leq \frac{\sum_{g \in G(C)} p(g)}{\sum_{g \in H} p(g)} - 1 = \frac{\sum_{g \in G(C) \setminus H} p(g)}{\sum_{g \in H} p(g)} \leq \frac{\sum_{g \in G(C) \setminus G^\eta(C)} p(g)}{\sum_{g \in G^\eta(C)} p(g)} < \eta L,$$

as desired. ■

## 4 Proof of the main results

We here prove Theorems 2 and 5. We let  $\varepsilon, \beta \in (0, 1)$  satisfy the assumptions of Theorem 2, and  $\hat{q}$  be another transition matrix over  $S$ . We assume that  $\hat{q}$  is  $(\varepsilon, \beta)$ -close to  $q$ .

## 4.1 On graphs

For every proper subset  $C$  of  $S$  and every  $\eta > 0$ , we denote by  $\widehat{G}^\eta(C)$  the set of  $\eta$ -maximal graphs under  $\widehat{q}$ . For every  $C$ -graph  $g$ ,  $\widehat{p}(g) = \prod_{s \in C} \widehat{q}(g(s) | s)$  is the weight of  $g$  under  $\widehat{q}$ .

**Lemma 11** *For every proper subset  $C$  of  $S$ ,*

$$\frac{1 - \beta}{|S|^2} \sum_{s \in C} \mu_s q(\overline{C} | s) \leq \sum_{s \in C} \mu_s \widehat{q}(\overline{C} | s) \leq (1 + \beta) |S|^2 \sum_{s \in C} \mu_s q(\overline{C} | s). \quad (9)$$

**Proof.** Let  $s_0 \in C$  and  $t_0 \in \overline{C}$  maximize the quantity  $\mu_s q(t | s)$  amongst  $s \in C$  and  $t \in \overline{C}$ . Then  $\mu_{s_0} q(t_0 | s_0) \geq \sum_{s \in C} \mu_s q(\overline{C} | s) / |S|^2 \geq \zeta_q / |S|^2 > \varepsilon \zeta_q$ . Since  $\widehat{q}$  is  $(\varepsilon, \beta)$ -close to  $q$ ,  $\widehat{q}(t_0 | s_0) \geq (1 - \beta) q(t_0 | s_0)$ . In particular,

$$\sum_{s \in C} \mu_s \widehat{q}(\overline{C} | s) \geq \mu_{s_0} \widehat{q}(t_0 | s_0) \geq (1 - \beta) \mu_{s_0} q(t_0 | s_0) \geq \frac{1 - \beta}{|S|^2} \sum_{s \in C} \mu_s q(\overline{C} | s), \quad (10)$$

and the left hand side inequality in (9) holds.

Let  $s_1 \in C$  and  $t_1 \in \overline{C}$  maximize the quantity  $\mu_s \widehat{q}(t | s)$  amongst  $s \in C$  and  $t \in \overline{C}$ . By (10),  $\mu_{s_1} \widehat{q}(t_1 | s_1) \geq \sum_{s \in C} \mu_s \widehat{q}(\overline{C} | s) / |S|^2 \geq (1 - \beta) \zeta_q / |S|^4 > \varepsilon$ . Since  $\widehat{q}$  is  $(\varepsilon, \beta)$ -close to  $q$ ,  $q(t_1 | s_1) \geq \widehat{q}(t_1 | s_1) / (1 + \beta)$ . Therefore

$$\sum_{s \in C} \mu_s q(\overline{C} | s) \geq \mu_{s_1} q(t_1 | s_1) \geq \frac{1}{1 + \beta} \mu_{s_1} \widehat{q}(t_1 | s_1) \geq \frac{1}{(1 + \beta) |S|^2} \sum_{s \in C} \mu_s \widehat{q}(\overline{C} | s),$$

and the right hand side inequality holds as well. ■

**Lemma 12** *Let  $C \subset S$  and  $s \in C$  be given. For every  $g \in G^\beta(C)$  (resp.  $g \in \widehat{G}^\beta(C)$ )  $\mu_s q(g(s) | s) \geq \varepsilon \zeta_q$  (resp.  $\mu_s \widehat{q}(g(s) | s) \geq \varepsilon \zeta_q$ ).*

Note that the second claim is not symmetric to the first, since in both we use the stationary distribution of  $q$ .

**Proof.** The proof is quite similar for  $g \in G^\beta(C)$  and  $g \in \widehat{G}^\beta(C)$ . We prove the lemma for the former, and mention where the proof for the latter differs.

Let  $g \in G^\beta(C)$  be arbitrary. The proof is by induction over the number of states in  $C$ .

If  $|C| = 1$ , then  $C = \{s\}$  for some  $s \in S$ . Since  $g$  is  $\beta$ -maximal,  $\mu_s q(g(s) \mid s) \geq \beta/|S| \mu_s q(\overline{C} \mid s) \geq \frac{\zeta_q}{|S|} \beta$  (for  $g \in \widehat{G}^\beta(C)$ , by Lemma 11,  $\mu_s \widehat{q}(g(s) \mid s) \geq \frac{\beta}{|S|} \mu_s \widehat{q}(\overline{C} \mid s) \geq \beta \frac{1-\beta}{|S|^3} \mu_s q(\overline{C} \mid s) \geq \beta \frac{1-\beta}{|S|^3} \zeta_q$ ).

Consider now the case  $|C| > 1$ .

We first assume that there are at least two edges of  $g$  whose endpoints do not belong to  $C$ . Let  $s_1 \neq s_2 \in C$ . Let  $g_i$  be the restriction of  $g$  to  $C \setminus \{s_i\}$ ,  $i = 1, 2$ . By **P1**,  $g_i \in G^\beta(C \setminus \{s_i\})$ . Since any edge of  $g$  is an edge of  $g_1$  or  $g_2$  (or both), the induction hypothesis applied to  $C \setminus \{s_i\}$  and  $g_i$ ,  $i = 1, 2$ , implies that the claim holds for  $g$ .

Assume now that there is a unique state  $s_1 \in C$  such that  $g(s_1) \notin C$ . Let  $g_1$  be the restriction of  $g$  to  $C \setminus \{s_1\}$ . By **P1**,  $g_1 \in G^\beta(C \setminus \{s_1\})$ . By the induction hypothesis applied to  $C \setminus \{s_1\}$  and  $g_1$ ,  $\mu_s q(g(s) \mid s) \geq \varepsilon \zeta_q$  for every  $s \in C \setminus \{s_1\}$ . Thus, it remains to show that  $\mu_{s_1} q(g(s_1) \mid s_1) \geq \varepsilon \zeta_q$ .

Let  $s_2 \in C$  maximize the quantity  $\mu_s q(\overline{C} \mid s)$  amongst  $s \in C$  (for  $g \in \widehat{G}^\beta(C)$ , it is chosen to maximize  $\mu_s \widehat{q}(\overline{C} \mid s)$ ). By the definition of  $\zeta_q$ ,  $\mu_{s_2} q(\overline{C} \mid s_2) \geq \zeta_q/|S|$  (for  $g \in \widehat{G}^\beta(C)$ , by Lemma 11,  $\mu_{s_2} \widehat{q}(\overline{C} \mid s_2) \geq (1 - \beta) \zeta_q/|S|^3$ ). Let  $\widehat{g} \in G^1(S \setminus C)$  (for  $g \in \widehat{G}^\beta(C)$ , one also chooses  $\widehat{g} \in G^1(S \setminus C)$ ). By **P0** and **P2**,  $\widehat{g} \cup g_1 \in G(S \setminus \{s_1\})$ .

Let  $\overline{g} \in G^1(S \setminus \{s_2\})$ . Since  $\overline{g}_{|S \setminus C}$  is a  $S \setminus C$ -graph, we have  $p(\widehat{g}) \geq p(\overline{g}_{|S \setminus C})$ . Since for every  $t \in \overline{C}$ ,  $\overline{g}_{|C \setminus \{s_2\}} \cup (s_2, t)$  is a  $C$ -graph,  $p(g) \geq p(\overline{g}_{|C \setminus \{s_2\}}) q(t \mid s_2)$ . In particular,  $p(\widehat{g}) p(g) \geq \beta p(\overline{g}) q(t \mid s_2)$  for every  $t \in \overline{C}$ , and therefore  $p(\widehat{g}) p(g) \geq \frac{\beta}{|S|} p(\overline{g}) q(\overline{C} \mid s_2)$ .

Denote  $\sum = \sum_{y \in S} \sum_{g \in G(S \setminus \{y\})} p(g)$ . By (6),  $\mu_s = \frac{1}{\sum} \sum_{g \in G(S \setminus \{s\})} p(g)$ . In particular,

$$\begin{aligned}
\frac{\zeta_q}{|S|} &\leq \mu_{s_2} q(\overline{C} \mid s_2) \leq \frac{\sum_{g \in G(S \setminus \{s_2\})} p(g)}{\sum} \times q(\overline{C} \mid s_2) \\
&\leq \frac{L p(\widehat{g}) q(\overline{C} \mid s_2)}{\sum} \leq \frac{L \times |S|}{\sum \beta} p(\widehat{g}) p(g) \\
&= \frac{L \times |S|}{\sum \beta} p(\widehat{g} \cup g \setminus (s_1, g(s_1))) q(g(s_1) \mid s_1) \\
&\leq \frac{L \times |S|}{\sum \beta} \sum_{g \in G(S \setminus \{s_1\})} p(g) \times q(g(s_1) \mid s_1) \\
&= \frac{L \times |S|}{\beta} \mu_{s_1} q(g(s_1) \mid s_1).
\end{aligned}$$

But then  $\mu_{s_1} q(g(s_1) \mid s_1) \geq \varepsilon \zeta_q$ , as desired. The calculation for  $g \in \widehat{G}^\beta(C)$  is analogous. ■

**Corollary 13** *For every proper subset  $C$  of  $S$ ,*

$$\left| 1 - \frac{\widehat{p}(g)}{p(g)} \right| \leq (|S| + 1)\beta, \text{ for every } g \in G^\beta(C) \cup \widehat{G}^\beta(C) \quad (11)$$

and

$$\left| \frac{\sum_{g \in H} \widehat{p}(g)}{\sum_{g \in H} p(g)} - 1 \right| < (|S| + 1)\beta, \text{ where } H = G^\beta(C) \cup \widehat{G}^\beta(C).$$

Thus, the weights of  $\beta$ -maximal graphs under  $q$  and  $\widehat{q}$  are close.

**Proof.** Note first that the second inequality follows immediately from the first one. Let us prove (11). Let  $g \in G^\beta(C)$ . By Lemma 12,  $\mu_s q(g(s) \mid s) \geq \varepsilon \zeta_q$  for every  $s \in C$ . Since  $\widehat{q}$  is  $(\varepsilon, \beta)$ -close to  $q$ ,  $(1 - \beta)q(g(s) \mid s) \leq \widehat{q}(g(s) \mid s) \leq (1 + \beta)q(g(s) \mid s)$ . Multiplying this inequality over  $s \in C$  yields  $(1 - \beta)^{|C|} p(g) \leq \widehat{p}(g) \leq (1 + \beta)^{|C|} p(g)$ , and (11) follows.

The proof for  $g \in \widehat{G}^\beta(C)$  is similar. ■

## 4.2 Proof of Theorem 2

**Proposition 14** *The transition matrix  $\widehat{q}$  is irreducible.*

**Proof.** It is enough to prove that for every non-empty subset  $C \subset S$ , there exists  $s \in C$ , and  $t \notin C$  such that  $\hat{q}(t | s) > 0$ .

Let  $s_1 \in C$  and  $t_1 \notin C$  be such that  $\mu_{s_1} q(t_1 | s_1) \geq \zeta_q / |S|^2 > \varepsilon \zeta_q$ . Since  $\hat{q}$  is  $(\varepsilon, \beta)$ -close to  $q$ ,  $\hat{q}(t_1 | s_1) > (1 - \beta)q(t_1 | s_1) > 0$ . ■

We need the following technical Lemma.

**Lemma 15** 1. Let  $(a_i)_{i=1}^I$  and  $(b_i)_{i=1}^I$  be positive numbers, and let  $\varepsilon > 0$ . If  $\left| \frac{a_i}{b_i} - 1 \right| < \varepsilon$  for every  $i = 1, \dots, I$  then  $\left| \frac{\sum_{i=1}^I a_i}{\sum_{i=1}^I b_i} - 1 \right| < \varepsilon$  and  $\left| \frac{\min\{a_1, a_2, \dots, a_I\}}{\min\{b_1, b_2, \dots, b_I\}} - 1 \right| < \varepsilon$ .

2. Let  $\varepsilon \in (0, 1/3)$ , and let  $a, A, b, B > 0$ . If  $\left| \frac{a}{b} - 1 \right| < \varepsilon$  and  $\left| \frac{b}{B} - 1 \right| < \varepsilon$  then  $\left| \frac{a/b}{A/B} - 1 \right| < 3\varepsilon$ .

**Proof.** The proof of the first part is left to the reader. For the second part, note that  $1/(1 + \varepsilon) < B/b < 1/(1 - \varepsilon)$ , which implies that  $B/b - 1 < \varepsilon/(1 - \varepsilon)$ . In particular,

$$\left| \frac{a/b}{A/B} - 1 \right| \leq \left( \left| \frac{a}{A} - 1 \right| + 1 \right) \left| \frac{B}{b} - 1 \right| + \left| \frac{a}{A} - 1 \right| < (1 + \varepsilon) \frac{\varepsilon}{1 - \varepsilon} + \varepsilon < 3\varepsilon.$$

■

**Proposition 16** For each  $s \in S$ ,

$$\left| 1 - \frac{\hat{\mu}_s}{\mu_s} \right| < 18\beta L.$$

**Proof.** Fix  $s \in S$ . By (6),

$$\mu_s = \frac{\sum_{G(S \setminus \{s\})} p(g)}{\sum_{y \in S} \sum_{G(S \setminus \{y\})} p(g)} \text{ and } \hat{\mu}_s = \frac{\sum_{G(S \setminus \{s\})} \hat{p}(g)}{\sum_{y \in S} \sum_{G(S \setminus \{y\})} \hat{p}(g)}.$$

For every  $y \in S$ , define  $H_y = G^\beta(S \setminus \{y\}) \cup \hat{G}^\beta(S \setminus \{y\})$ . Define

$$\mu'_s = \frac{\sum_{H_s} p(g)}{\sum_{y \in S} \sum_{H_y} p(g)} \text{ and } \hat{\mu}'_s = \frac{\sum_{H_s} \hat{p}(g)}{\sum_{y \in S} \sum_{H_y} \hat{p}(g)}.$$

By Lemma 10 and Lemma 15,  $\left| \frac{\mu_s}{\mu'_s} - 1 \right| < 3\beta L$  and  $\left| \frac{\hat{\mu}_s}{\hat{\mu}'_s} - 1 \right| < 3\beta L$ . By Lemmas 10 and 15,  $\left| \frac{\hat{\mu}'_s}{\mu'_s} - 1 \right| < 3(|S| + 1)\beta$ . Since  $L \geq |S| \geq 2$ , the result follows by Lemma 15(2). ■



### 4.3 Proof of Theorem 5

**Proposition 17** *For every proper subset  $C$  of  $S$ , every  $s \in C$  and  $t \notin C$ ,*

$$|\mathbf{Q}_{s,q}(t | C) - \mathbf{Q}_{s,\hat{q}}(t | C)| < 12\beta L.$$

**Proof.** Denote  $H = G^\beta(C) \cup \hat{G}^\beta(C)$ , and  $H_{s,t} = H \cap G_{s,t}(C)$ .

Assume first that  $H_{s,t} \neq \emptyset$ . By (8), one has

$$\left| \mathbf{Q}_{s,q}(t|C) - \frac{\sum_{H \cap G_{s,t}(C)} p(g)}{\sum_{G(C)} p(g)} \right| \leq \beta L.$$

Since  $\left| \frac{\sum_H p(g)}{\sum_{G(C)} p(g)} - 1 \right| \leq \beta L$ , this yields, by Lemma 15(2),

$$\left| \mathbf{Q}_{s,q}(t|C) - \frac{\sum_{H \cap G_{s,t}(C)} p(g)}{\sum_H p(g)} \right| \leq \beta L + 3\beta L \leq 4\beta L, \quad (12)$$

and a similar inequality holds with  $q$  replaced by  $\hat{q}$ .

By Corollary 13 and Lemma 15(1),

$$\left| \frac{\sum_{H \cap G_{s,t}(C)} \hat{p}(g)}{\sum_{H \cap G_{s,t}(C)} p(g)} - 1 \right| \leq \beta(|S| + 1) \text{ and } \left| \frac{\sum_H \hat{p}(g)}{\sum_H p(g)} \right| \leq \beta(|S| + 1). \quad (13)$$

By Lemma 15(2), (13) implies

$$\left| \frac{\sum_{H \cap G_{s,t}(C)} p(g)}{\sum_H p(g)} - \frac{\sum_{H \cap G_{s,t}(C)} \hat{p}(g)}{\sum_H \hat{p}(g)} \right| \leq 3\beta(|S| + 1),$$

which implies, using (12),

$$|\mathbf{Q}_{s,q}(t|C) - \mathbf{Q}_{s,\hat{q}}(t|C)| \leq \beta(8L + 3(|S| + 1)).$$

If, on the other hand,  $H_{s,t} = \emptyset$ , then by (8) and the definition of  $H_{s,t}$ ,  $\mathbf{Q}_{s,q}(t | C), \mathbf{Q}_{s,\hat{q}}(t | C) \leq \beta L$ . ■

**Proposition 18** *For every proper subset  $C$  of  $S$ ,*

$$\frac{1}{2|S|^2} K_C \leq \hat{K}_C \leq 2|S|^2 K_C.$$

**Proof.** We first argue that

$$K_C = \frac{\sum_{s \in C} \mu_s}{\sum_{s \in C} \mu_s q(\bar{C} | s)}. \quad (14)$$

Indeed, define the r.v.  $\rho_n$  as the average length of visits to  $C$  that end before stage  $n + 1$ :

$$\rho_n = \frac{\sum_{p=1}^n 1_{s_p \in C}}{\sum_{p=1}^n 1_{s_p \in C} 1_{s_{p+1} \notin C}}. \quad (15)$$

By the ergodic theorem, the sequence  $(\rho_n)$  converges,  $\mathbf{P}_{s_1, q}$ -a.s. to  $K_C$ , while the right hand side in (15) converges  $\mathbf{P}_{s_1, q}$ -a.s. to  $\frac{\sum_{s \in C} \mu_s}{\sum_{s \in C} \mu_s q(\bar{C} | s)}$ . The identity (14) follows.

By Proposition 16, for every  $s \in C$ ,

$$(1 - 18\beta L)\mu_s < \hat{\mu}_s < (1 + 18\beta L)\mu_s. \quad (16)$$

By Lemma 11,

$$\frac{1 - \beta}{|S|^2} \sum_{s \in C} \mu_s q(\bar{C} | s) \leq \sum_{s \in C} \mu_s \hat{q}(\bar{C} | s) \leq (1 + \beta)|S|^2 \sum_{s \in C} \mu_s q(\bar{C} | s). \quad (17)$$

Eqs. (16) and (17) yield

$$\frac{(1 - 18\beta L)(1 - \beta)}{|S|^2} \sum_{s \in C} \mu_s q(\bar{C} | s) \leq \sum_{s \in C} \hat{\mu}_s \hat{q}(\bar{C} | s) \leq (1 + \beta)(1 + 18\beta L)|S|^2 \sum_{s \in C} \mu_s q(\bar{C} | s). \quad (18)$$

Summing up Eq. (16) over  $s \in C$  gives

$$(1 - 18\beta L) \sum_{s \in C} \mu_s < \sum_{s \in C} \hat{\mu}_s < (1 + 18\beta L) \sum_{s \in C} \mu_s. \quad (19)$$

The Proposition follows by dividing (19) by (18). ■

## 5 Proof of the variations

We here prove Theorems 4 and 6. We shall follow the previous proofs, and will point out which changes are needed. We let  $a, \varepsilon, \beta$  be given, that satisfy the assumptions of Theorem 4. The result of Section 4.1 still hold for every proper subset  $C$  of  $S_1$ , namely Lemmas 11, 12 and Corollary 13 are still valid, provided the assumption  $C \subset S$  is replaced by the assumption  $C \subset S_1$ .

## 5.1 Proof of Theorem 4

We need the following observation.

**Lemma 19** *For every  $y \in S_1$ , there exists a  $(a/L)^{|S|}$ -maximal graph  $\bar{g} \in G(S_1 \setminus y)$  such that all paths of  $\bar{g}$  lead to  $y$ .*

**Proof.** By **P2**, for every  $s \in S_1 \setminus y$  there is a  $\frac{a}{L}$ -maximal  $S_1 \setminus y$ -graph  $g_s$  in which  $s$  leads to a state in  $y$ . Let  $h_s$  be the path in  $g_s$  that connects  $s$  to  $y$  (this is a set of edges). Let  $\bar{g}$  be a  $S_1 \setminus y$ -graph that is contained in  $\cup_{s \in S_1 \setminus y} h_s$ . Then  $\bar{g}$  satisfies the conditions. ■

We next prove the two assertions of Theorem 4.

**Lemma 20** *All states of  $S_1$  belong to the same recurrent set for  $\hat{q}$ .*

**Proof.** It is enough to prove that for each  $C \subset S_1$ , there exists  $s \in C$  and  $t \in \bar{C}$  such that  $\hat{q}(t|s) > 0$ . The proof of Proposition 14 still applies, provided  $\zeta_q$  is replaced by  $\zeta_q^1$ . ■

**Lemma 21** *For each  $s \in S_1$ ,*

$$\left| 1 - \frac{\hat{\mu}(s|S_1)}{\mu(s|S_1)} \right| \leq 18\beta L.$$

**Proof.** The proof goes essentially as in Proposition 16. Set  $\eta = \beta/(a/L)^{|S|} < (a/L)^S$ , and fix  $s \in S_1$ . By (6),

$$\mu(s|S_1) = \frac{\sum_{G(S \setminus \{s\})} p(g)}{\sum_{y \in S_1} \sum_{G(S \setminus \{y\})} p(g)} \text{ and } \hat{\mu}(s|S_1) = \frac{\sum_{G(S \setminus \{s\})} \hat{p}(g)}{\sum_{y \in S_1} \sum_{G(S \setminus \{y\})} \hat{p}(g)}.$$

For every  $y \in S_1$ , define  $H_y = G^\eta(S \setminus \{y\}) \cup \hat{G}^\eta(S \setminus \{y\})$ . Define

$$\mu'(s|S_1) = \frac{\sum_{H_s} p(g)}{\sum_{y \in S_1} \sum_{H_y} p(g)} \text{ and } \hat{\mu}'(s|S_1) = \frac{\sum_{H_s} \hat{p}(g)}{\sum_{y \in S_1} \sum_{H_y} \hat{p}(g)}.$$

Fix for a moment  $y \in S_1$ . By Lemma 19 there is a  $(a/L)^{|S|}$ -maximal  $S_1 \setminus \{y\}$ -graph  $\bar{g}$  such that all its paths lead to  $y$ . Let  $g \in G^\eta(S \setminus \{y\})$ , and  $g_{S_1 \setminus \{y\}}$ ,  $g_{S \setminus S_1}$  its restrictions to  $S_1 \setminus \{y\}$  and  $S \setminus S_1$ . Using the above remark, the

graph  $\bar{g} \cup g_{S \setminus S_1}$  is a  $S \setminus \{y\}$ -graph. Therefore,  $g_{S_1 \setminus \{y\}}$  is  $\eta(a/L)^{|S|}$ -maximal ( $= \beta$ -maximal). By Corollary 13 one has  $\left|1 - \frac{\hat{p}(g_{S_1 \setminus \{y\}})}{p(g_{S_1 \setminus \{y\}})}\right| < (|S| + 1)\beta$ . Since  $q$  and  $\hat{q}$  coincide outside  $S_1$ ,  $p(g_{S \setminus S_1}) = \hat{p}(g_{S \setminus S_1})$ . Thus,  $\left|1 - \frac{\hat{p}(g)}{p(g)}\right| < (|S| + 1)\beta$ . Lemma 10 and Lemma 15 implies that  $\left|\frac{\mu(s|S_1)}{\mu'(s|S_1)} - 1\right| < 3\beta L$  and  $\left|\frac{\hat{\mu}(s|S_1)}{\hat{\mu}'(s|S_1)} - 1\right| < 3\beta L$ . By Lemmas 10 and 15,  $\left|\frac{\hat{\mu}'(s|S_1)}{\mu'(s|S_1)} - 1\right| < 3(|S| + 1)\beta$ . Since  $L \geq |S| \geq 2$ , the Lemma follows by Lemma 15(2). ■

## 5.2 Proof of Theorem 6

The proof of the first two assertions in Theorem 6 is identical to the proof of the two assertions in Theorem 5 (see Propositions 17 and 18). We omit it. We now prove a slightly strengthened version of the last assertion.

**Proposition 22** *Let  $\eta \leq \varepsilon \zeta_q^1$  be such that  $\left|1 - \frac{\hat{q}(t|s)}{q(t|s)}\right| \leq \beta$  whenever  $\mu_s \max(q(t|s), \hat{q}(t|s)) \geq \eta$ . One has*

$$\frac{1}{c}K_{S_1} \leq \hat{K}_{S_1} \leq cK_{S_1} \text{ or } K_{S_1}, \hat{K}_{S_1} \geq \frac{1}{2|S|} \times \frac{\mu_{S_1}}{\eta}.$$

The last statement in Theorem 6 corresponds to the case  $\eta = \varepsilon \zeta_q^1$ .

**Proof.** Fix  $s \in S_1$ . By (14),

$$K_{S_1} = \frac{1}{\sum_{t \in S_1} \mu(t|S_1)q(\bar{S}_1 | t)},$$

and a similar equality holds for  $\hat{K}_{S_1}$ , involving  $\hat{\mu}$  and  $\hat{q}$ . By Theorem 4(2) and Lemma 15, the ratio between  $\hat{K}_{S_1}$  and  $\frac{1}{\sum_{t \in S_1} \mu(t|S_1)\hat{q}(\bar{S}_1 | t)}$  is between  $1 - 54\beta L$  and  $1 + 54\beta L$ .

If for every  $t \in S_1$  and  $u \notin S_1$ ,  $\mu_t q(u | t) < \eta$  and  $\mu_t \hat{q}(u | t) < \eta$ , then  $K_{S_1} \geq \frac{\mu_{S_1}}{|S|^2 \eta}$  and  $\hat{K}_{S_1} \geq (1 - 54\beta L) \times \frac{\mu_{S_1}}{|S|^2 \eta}$ , as desired.

If, on the other hand, there exist  $t \in S_1$  and  $u \notin S_1$  such that  $\mu_t q(u | t) \geq \eta$  or  $\mu_t \hat{q}(u | t) \geq \eta$  then  $\left|1 - \frac{\hat{q}(u|t)}{q(u|t)}\right| \leq \beta$ , and therefore  $\mu_t q(u | t) \geq (1 - \beta)\eta$  and  $\mu_t \hat{q}(u | t) \geq (1 - \beta)\eta$ . For every  $t \in S_1$  and  $u \notin S_1$  such that  $\mu_t q(u | t) < \eta$  and  $\mu_t \hat{q}(u | t) < \eta$  we have  $\mu_t q(u | t) \leq \sum_{t \in S_1} \mu_t q(\bar{S}_1 | t)$  and  $\mu_t \hat{q}(u | t) \leq \sum_{t \in S_1} \mu_t \hat{q}(\bar{S}_1 | t)$ . It follows that the ratio between  $\sum_{t \in S_1} \mu_t q(\bar{S}_1 | t)$  and  $\sum_{t \in S_1} \mu_t \hat{q}(\bar{S}_1 | t)$  is at most  $|S|^2$ . The result follows. ■